

BOX REPRESENTATIONS OF EMBEDDED GRAPHS

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ABSTRACT. A d -box is the cartesian product of d intervals of \mathbb{R} and a d -box representation of a graph G is a representation of G as the intersection graph of a set of d -boxes in \mathbb{R}^d . It was proved by Thomassen in 1986 that every planar graph has a 3-box representation. In this paper we prove that every graph embedded in a fixed orientable surface, without short non-contractible cycles, has a 5-box representation. This directly implies that there is a function f , such that in every graph of genus g , a set of at most $f(g)$ vertices can be removed so that the resulting graph has a 5-box representation. We show that such a function f can be made linear in g . Finally, we prove that for any proper minor-closed class \mathcal{F} , there is a constant $c(\mathcal{F})$ such that every graph of \mathcal{F} without cycles of length less than $c(\mathcal{F})$ has a 3-box representation, which is best possible.

1. INTRODUCTION

For $d \geq 1$, a d -box is the cartesian product $I_1 \times I_2 \times \cdots \times I_d$ of d intervals of \mathbb{R} . A d -box representation of a graph $G = (V, E)$ is a collection $\mathcal{R} = (B_v)_{v \in V}$ of d -boxes in \mathbb{R}^d such that any two boxes B_u and B_v intersect if and only if the corresponding vertices u and v are adjacent in G . In other words, G is the intersection graph of the boxes $(B_v)_{v \in V}$. The *boxicity* of a graph G , denoted by $\text{box}(G)$ and introduced by Roberts in 1969 [11], is the smallest integer d such that G has a d -box representation.

It was proved by Thomassen in 1986 that planar graphs have boxicity at most 3 [13], which is best possible (as shown by the planar graph obtained from a complete graph on 6 vertices by removing a perfect matching [11]). It is natural to investigate how this result on planar graphs extends to graphs embeddable on surfaces of higher genus. Let $\text{box}(g)$ be the supremum of the boxicity of all graphs embeddable in a surface of Euler genus g . The result of Thomassen on planar graphs was extended in [3] by showing that for any $g \geq 0$, $\text{box}(g) \leq 5g + 3$ (prior to this result, it was not known whether $\text{box}(g)$ was finite). In [2], we proved the existence of two constants $c_1, c_2 > 0$, such that $c_1 \sqrt{g \log g} \leq \text{box}(g) \leq c_2 \sqrt{g \log g}$. The proof of the upper bound relies on a connection between boxicity and acyclic coloring established in [3]. It was noted there that using this connection and a result of Kawarabayashi and Mohar [6], it could be proved that there is a function f such that any graph embedded in a surface of Euler genus g , such that all non-contractible cycles have length at least $f(g)$, has boxicity at most 42. The main result of this paper is to reduce this bound to 5 for orientable surfaces (Theorem 8). Similar ideas are then used to show that toroidal graphs have boxicity at most 6, and toroidal graphs without non-contractible triangles have boxicity at most 5

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(Theorem 10). This improves on [3], where it was proved that toroidal graphs have boxicity at most 7, while there are toroidal graphs of boxicity 4.

An immediate consequence of Theorem 8 is that if G has genus g , then a set of at most $\sum_{i=1}^g f(i)$ vertices can be removed in G so that the resulting graph has boxicity at most 5. However, the bound we obtain for $f(g)$ in Theorem 8 is exponential in g . We show the following improvement: if G is embedded in a surface of Euler genus $g > 0$, then a set of at most $60g - 30$ vertices can be removed in G so that the resulting graph has boxicity at most 5 (Theorem 13). Note that this result is proved for any surface, orientable or not.

In [3], it was proved that there is a function c such that if G is embedded in a surface of Euler genus g , and has no cycle of length less than $c(g)$, then G has boxicity at most 4. Here, we show there is a function c such that if G has no K_t -minor and no cycle of length less than $c(t)$, then G has boxicity at most 3 (Corollary 15). This is best possible already for $t = 6$. This result follows from a more general theorem on path-degenerate graphs (Theorem 14). This general result also implies, together with earlier results from [5], that there is a constant c such that any graph embeddable in a surface of Euler genus g , with no cycle of length less than $c \log g$, has boxicity at most 3 (Corollary 16). This is best possible up to the choice of the constant c .

Some of the results we will use originate from the proof of Thomassen [13] that planar graphs have boxicity at most 3, without being explicitly stated there. In Section 2, we explain how these results can be derived from [13], which might be of independent interest. We will then prove the main results of this paper in Sections 3, 4, and 5. In the remainder of this section, we review the necessary background on boxicity and graphs on surfaces. We then give a simple proof that embedded graphs of large edge-width have boxicity at most 7 (this will be improved to 5 in Section 3).

Boxicity. Let $G = (V, E)$ be a graph, and let $\mathcal{R} = (B_v)_{v \in V}$ be a d -box representation of G . For a d -box $B_v = I_1 \times I_2 \times \cdots \times I_d$, and an integer $1 \leq i \leq d$, we refer to I_i as *the i -th interval of v* . For $1 \leq i \leq d$, let \mathcal{I}_i be the interval representation consisting of all i -th intervals of the vertices of G in \mathcal{R} . Each interval representation \mathcal{I}_i , $1 \leq i \leq d$, corresponds to an interval graph G_i with vertex-set V . Observe that each graph G_i is a supergraph of G , and any two d -boxes B_u and B_v intersect if and only if for all $1 \leq i \leq d$ the intervals corresponding to u and v intersect in \mathcal{I}_i , or equivalently, if the vertices u and v are adjacent in the interval graph G_i .

For two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex-set V , the *intersection* $G_1 \cap G_2$ of G_1 and G_2 is defined as the graph $G = (V, E_1 \cap E_2)$. The discussion above implies that the boxicity of a graph G can be equivalently defined as the least k such that G can be expressed as the intersection of k interval graphs on the same vertex-set. In this paper, the two definitions will be used and we will often switch from one to the other, depending of the situation, i.e whenever we consider some d -box representation $\mathcal{R} = (B_v)_{v \in V}$ we will implicitly consider it as a representation $\mathcal{R} = (\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_d)$ as defined in the previous paragraph, and vice-versa. If \mathcal{R} is a representation of G , we will often say that \mathcal{R} *represents* G , or *induces* G .

Graphs on surfaces. We refer the reader to the book by Mohar and Thomassen [9] for more details or any notion not defined here. All the graphs in this paper are simple (i.e., without loops and multiple edges). A *surface* is a non-null compact connected 2-manifold without boundary. A surface can be orientable or non-orientable. The *orientable surface* \mathbb{S}_h of genus h is obtained by adding $h \geq 0$ *handles* to the sphere; while the *non-orientable surface* \mathbb{N}_k of genus k is formed by adding $k \geq 1$ *cross-caps* to the sphere. The *Euler genus* of a surface Σ is defined as twice its genus if Σ is orientable, and as its non-orientable genus otherwise.

We say that an embedding is *cellular* if every face is homeomorphic to an open disk of \mathbb{R}^2 . Using the fact that the boxicity of a graph G is at most k if and only if all the connected components of G have boxicity at most k , we will always be able to assume in this paper (using [9, Propositions 3.4.1 and 3.4.2]) that the considered embeddings are cellular (and we will do so implicitly).

Let G be a graph embedded in a surface of Euler genus $g > 0$. The *edge-width* of G is defined as the length of a smallest non-contractible cycle of G , and the *face-width* of G is defined as the minimum number of points of G intersected by a non-contractible curve on the surface. Given a 2-sided cycle C in G , and an orientation of C , the set of neighbors of C incident to an edge leaving the left side of C is denoted by $L(C)$, and the set of neighbors of C incident to an edge leaving the right side of C is denoted by $R(C)$ (see Figure 1, left, for an example in a triangulation). Note that $L(C)$ and $R(C)$ might not be disjoint. Given two non-contractible cycles C_1 and C_2 , $\text{dist}(C_1, C_2)$ denotes the minimum distance between a vertex of C_1 and a vertex of C_2 in G . If C is a non-contractible 2-sided cycle, we also define $\text{dist}(C, C)$ as the minimum length (number of edges) of a path starting with an edge incident to C and $L(C)$, and ending with an edge incident to C and $R(C)$ (note that $\text{dist}(C, C)$ does not depend of the chosen orientation of C).

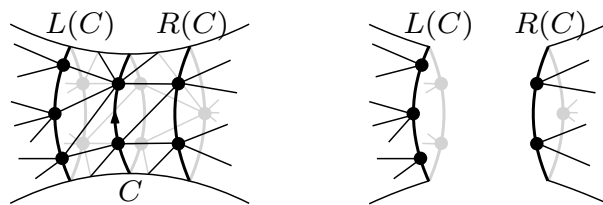


FIGURE 1. The sets $L(C)$ and $R(C)$ in a triangulation (left) and the embedded graph resulting from the removal of C (right).

Let G be a triangulation of \mathbb{S}_g . A collection of pairwise disjoint non-contractible cycles C_1, \dots, C_g of G is *planarizing* if the graph obtained from G by removing the vertices of C_1, \dots, C_g is planar. If for any $i \leq j$, $\text{dist}(C_i, C_j) \geq d$, then the planarizing collection C_1, \dots, C_g is said to have *minimum distance* at least d .

The following was proved by Thomassen [14]:

Theorem 1. [14] *Let d, g be integers, and let G be a triangulation of \mathbb{S}_g of edge-width at least $8(d+1)(2^g - 1)$. Then G contains a planarizing collection of induced cycles with minimum distance at least d .*

Triangulations of large edge-width have boxicity at most 7. To give a taste of the proofs of the main results in this paper, we now prove that *if $G = (V, E)$ is a triangulation of \mathbb{S}_g , $g \geq 1$, with edge-width at least $40(2^g - 1)$, then G has boxicity at most 7.*

Let C_1, \dots, C_g be a planarizing collection of induced cycles with minimum distance at least 4 (whose existence follows from Theorem 1) in G . We denote by C the set of vertices lying in one of the C_i 's. We also denote by N the set of vertices not in C with at least one neighbor in C , and set $R = V \setminus (C \cup N)$.

Since the collection C_1, \dots, C_g is planarizing, the set $V \setminus C = N \cup R$ induces a planar graph and therefore has a 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ by the result of Thomassen [13]. We extend this representation to C by mapping all the vertices of C to a big 3-box containing all the other 3-boxes of the representation. Since the distance in G between any two cycles C_i, C_j is at least 4, there is no edge between a neighbor of C_i and a neighbor of C_j . As a consequence, the set $C \cup N$ induces a disjoint union of g planar graphs (and is therefore planar). Let $(\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6)$ be a 3-box representation of this graph. We extend this representation to R by mapping all the vertices of R to a big 3-box containing all the other 3-boxes of the representation $(\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6)$. Finally, let \mathcal{I}_7 be the interval graph mapping all the vertices of C to $\{0\}$, all the vertices of N to $[0, 1]$, and all the vertices of R to $\{1\}$. We now prove that $\mathcal{R} = (\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_7)$ induces G . Each interval graph \mathcal{I}_i is clearly a supergraph of G , so we only need to show that every pair of non-adjacent vertices u, v in G is non-adjacent in at least one of the \mathcal{I}_i 's. This follows from the definition of $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ if $u, v \in N \cup R$, and from the definition of $(\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6)$ if $u, v \in C \cup N$. Finally if $u \in C$ and $v \in R$ then they are non-adjacent in \mathcal{I}_7 , as desired. \square

In Section 3 we will show that this argument can be applied to any embedded graph (not only triangulations) and that the bound on the boxicity can be improved to 5. The idea will be to group $\mathcal{I}_3, \mathcal{I}_6$, and \mathcal{I}_7 in a single interval graph. But for this we first need to see how much flexibility we have in choosing 3-box representations for planar graphs, which is the topic of the next section.

2. PLANAR GRAPHS

Recall that Thomassen [13] proved that planar graphs have boxicity at most 3. He actually proved significantly stronger results, which we will need here.

A d -box is *non-degenerate* if it is the cartesian product of d intervals of positive length. A representation of a graph G as the intersection of d -boxes is *strict* if (1) the boxes are non-degenerate, (2) the interiors of the boxes are pairwise disjoint, and (3) the intersection of any two boxes is a non-degenerate $(d - 1)$ -box.

Theorem 2. [13] *A graph G has a strict 2-box representation if and only if G is a proper subgraph of a 4-connected planar triangulation.*

Let G be a graph embedded in some surface. A triangle of G is said to be *facial* if it bounds a face of G . We will use the following immediate corollary of Theorem 2:

Corollary 3. *Let G be a plane graph such that all the triangles of G are facial. If G is not a triangulation, then G has a strict 2-box representation.*

Proof. Let H be the graph obtained from G by doing the following, for every non-triangular face f of G : add a cycle C_f of length $d(f)$ (the number of edges in a boundary walk of f) inside f , join one vertex of C_f to each of the other vertices of C_f (with the newly added edges staying inside the region bounded by C_f), and then join each vertex of C_f to two consecutive vertices in a boundary walk of f . This can be done in such a way that H is a (simple) triangulation. Since all the triangles of G are facial, H is a triangulation without separating triangle. Observe that H contains at least 5 vertices, so H is a 4-connected triangulation. The result now follows directly from Theorem 2. \square

Consider a strict 3-box representation $(B_v)_{v \in V}$ of a planar graph $G = (V, E)$, and let uvw be a triangle of G . Then uvw has an *empty inner corner* in the representation if there is a small 3-box C , whose interior is disjoint from the interior of all the boxes $(B_v)_{v \in V}$, such that the following holds: there is a corner c of C , and three faces f_u, f_v, f_w of C containing c , such that $C \cap B_u = f_u$, $C \cap B_v = f_v$, and $C \cap B_w = f_w$ (see Figure 2). In particular $c = f_u \cap f_v \cap f_w \in B_u \cap B_v \cap B_w$. Note that if a triangle has an empty inner corner, then the corresponding 3-box C defined above can be made arbitrarily small.

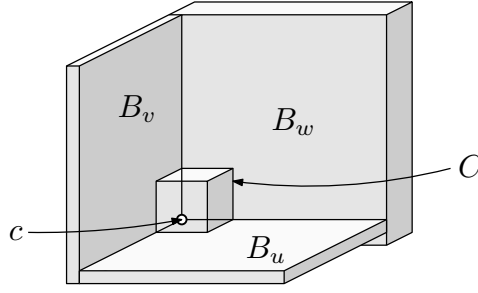


FIGURE 2. An empty inner corner in a strict 3-box representation.

We say that three intervals I_1, I_2, I_3 are *strictly overlapping* if $I_1 \cap I_2 \cap I_3$ is an interval of positive length, and none of the three intervals is contained in another one of the three.

The following simple lemma about empty inner corners will be particularly useful.

Lemma 4. *Let $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ be a strict 3-box representation of a planar graph G such that $(\mathcal{I}_1, \mathcal{I}_2)$ is a strict 2-box representation of G . Then any triangle uvw of G such that the intervals of u, v, w are strictly overlapping in \mathcal{I}_3 has an empty inner corner in $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$.*

Proof. Let uvw be a triangle of G . Without loss of generality, in $(\mathcal{I}_1, \mathcal{I}_2)$, u, v, w are mapped to rectangles R_u, R_v, R_w such that the point $(c_1, c_2) = R_u \cap R_v \cap R_w$ is a corner of R_v and R_w , but lies on a side of R_u (see Figure 3, left). Moreover, no rectangle of $(\mathcal{I}_1, \mathcal{I}_2)$ other than R_u, R_v, R_w contains (c_1, c_2) . Let I_u, I_v, I_w be the intervals corresponding to u, v, w in \mathcal{I}_3 . Since I_u, I_v, I_w are strictly overlapping, $I_u \cap I_v \cap I_w$ is an interval of positive length, and none of the three intervals is contained in another one. Let $[s, t] = I_v \cap I_w$. Observe that the interior of I_u intersects at least one of s, t (say t by symmetry), otherwise I_u would not intersect I_v or I_w (in an interval of positive length), or would be contained in I_v or I_w . It is then easy to check that the triangle uvw has an empty inner corner in $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$, touching the point with coordinate (c_1, c_2, t) (see Figure 3, right). \square

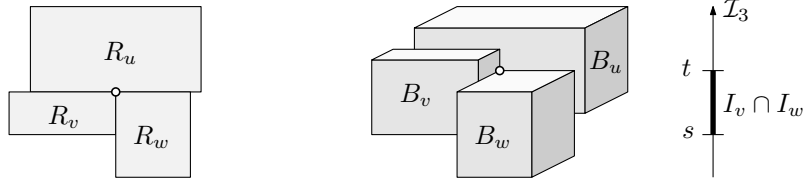


FIGURE 3. The 2-boxes of a triangle uvw in a strict 2-box representation (left), and an empty inner corner (depicted by a white dot) in the corresponding strict 3-box representation if the intervals of the third dimension are strictly overlapping (right). For the sake of readability, we write B_x instead of $R_x \times I_x$.

The following result was proved by Thomassen (see Theorem 4.5 in [13]), using a strong variant of Theorem 2.

Theorem 5. [13] *Every planar triangulation G has a strict 3-box representation such that every facial triangle of G , except one prescribed triangle, has an empty inner corner.*

The proof of Thomassen [13] indeed shows a slightly stronger result, which will be extremely useful in this paper.

Theorem 6. [13] *Let G be a planar triangulation G , with outerface uvw , and let B_u, B_v, B_w be any strict 3-box representation of u, v, w such that uvw has an empty inner corner C . Then B_u, B_v, B_w extends to a strict 3-box representation \mathcal{R} of G such that every facial triangle of G , except possibly uvw , has an empty inner corner in \mathcal{R} . Moreover, all 3-boxes except that of u, v, w are included in C .*

Let G be a graph embedded in the plane, or in \mathbb{S}_g with $g > 0$, and such that all the triangles are contractible. Then each triangle bounds a region homeomorphic to an open disk. This region is called the *interior* of the triangle. Let H be the graph obtained from G by removing (the vertices in) the interior of each triangle (distinct from the outerface of G , in case G is embedded in the plane). The graph H is called the *frame* of G . Note that H is embedded in the plane or \mathbb{S}_g with $g > 0$, with the property that all its triangles are facial. Moreover, if G is embedded in the plane, the outerfaces of G and H coincide.

A triangle in a graph G embedded in the plane is said to be *internal* if it is distinct from the outerface of G . The following direct consequence of Theorem 6 will be repeatedly used in the next section (in conjunction with Corollary 3 and Lemma 4).

Corollary 7. *Let G be a planar graph and let H be its frame. Then any strict 3-box representation of H in which each internal triangle has a fixed empty inner corner can be extended to a strict 3-box representation of G . Moreover, if uvw is an internal triangle of H and x is a vertex of G inside the region bounded by uvw in G , then x is mapped to a 3-box lying in the empty inner corner of uvw .*

3. LOCALLY PLANAR GRAPHS AND TOROIDAL GRAPHS

We are now ready to prove the main result of this paper.

Theorem 8. *Let G be a graph embedded in \mathbb{S}_g , $g \geq 1$, with edge-width at least $40(2^g - 1)$. Then G has boxicity at most 5.*

Proof. It is well-known that if G has edge-width at least k , then G is an induced subgraph of a triangulation of \mathbb{S}_g of edge-width at least k (see for instance Lemma 3.4 in [14]). Since having boxicity at most k is a property that is closed under taking induced subgraphs, in the following we can assume that G is a triangulation of \mathbb{S}_g . Since G has edge-width at least $40(2^g - 1) > 3$, all the triangles of G are contractible. Let H be the frame of G . Observe that H is a triangulation of \mathbb{S}_g of edge-width at least $40(2^g - 1)$. By Theorem 1, H has a collection of planarizing cycles C_1, \dots, C_g with minimum distance at least 4.

Consider the following 5 sets of vertices of G , forming a partition of $V(G)$:

- C : the union of all cycles C_1, \dots, C_g ,
- N : the union of all neighbors in H of a vertex of C ,
- R : the vertices of H not in $C \cup N$,
- T_1 : the vertices of G lying inside a non-facial triangle that does not intersect C ,
- T_2 : the vertices of G lying inside a non-facial triangle intersecting C .

Observe that since all the triangles of G are contractible, all the triangles of H are facial (and moreover, the triangles of H are precisely the faces of H , since H is a triangulation). Let H_1 be the planar graph obtained from H by deleting the vertices of C (i.e. H_1 is the subgraph of G induced by $N \cup R$). All the triangles of H_1 are facial, and we claim that H_1 is not a triangulation. To see this, observe that the deletion of each C_i produces two faces with vertex-set $L(C_i)$ and $R(C_i)$, respectively (see Figure 1, right). The two cycles bounding these faces in H_1 correspond to two cycles of H that are (freely) homotopic to C_i , and therefore non-contractible in \mathbb{S}_g . Since H has edge-width at least 4, the boundary walks of the two faces of H_1 resulting from the deletion of C_i have length at least 4, and H_1 is not a triangulation. By Corollary 3, H_1 has a strict 2-box representation $(\mathcal{I}_1, \mathcal{I}_2)$. We extend this 2-dimensional representation of H_1 to $C \cup T_2$ by mapping all the vertices of $C \cup T_2$ to a large 2-box containing all the 2-boxes of $(\mathcal{I}_1, \mathcal{I}_2)$. The images of the vertices of T_1 in $(\mathcal{I}_1, \mathcal{I}_2)$ will be defined later in the proof.

Let H_2 be the subgraph of H induced by $C \cup N$. Since the collection C_1, \dots, C_g has minimum distance at least 4, H_2 is the disjoint union of g graphs, each being a subgraph of H induced by C_i together with its neighborhood $N_H(C_i)$, for $1 \leq i \leq g$. As each of these graphs is embedded in a cylinder, they are all planar and so is H_2 . As before, we can argue that all the triangles of H_2 are facial, and H_2 is not a triangulation. It follows from Corollary 3 that H_2 has a strict 2-box representation $(\mathcal{I}_3, \mathcal{I}_4)$. We extend this 2-dimensional representation of H_2 to $R \cup T_1$ by mapping all the vertices of $R \cup T_1$ to a large 2-box containing all the 2-boxes of $(\mathcal{I}_3, \mathcal{I}_4)$. Again, the images of the vertices of T_2 in $(\mathcal{I}_3, \mathcal{I}_4)$ will be defined later.

For each vertex v of H , we choose a real $0 < \epsilon_v < \frac{1}{4}$, in such way that all the chosen ϵ are distinct. Let \mathcal{I}_5 be the following interval graph: each vertex v of C is mapped to $[\epsilon_v, 2 + \epsilon_v]$, each vertex v of N is mapped to $[1 + \epsilon_v, 4 + \epsilon_v]$, and each vertex v of R is mapped to $[3 + \epsilon_v, 5 + \epsilon_v]$. Observe that in the corresponding interval graph, C , N and R are cliques, N is complete to C and R , while there are no edges between C and R .

In order to complete the description of our 5-box representation $(\mathcal{I}_1, \dots, \mathcal{I}_5)$ of G , we only need to prescribe the images of the vertices of T_1 in $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5)$ and the images of the vertices of T_2 in $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$. All these images will be defined using Corollary 7, as we now explain.

Consider first the 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5)$, restricted to the vertices of H_1 (i.e. the vertices of $N \cup R$). Since $(\mathcal{I}_1, \mathcal{I}_2)$ is a strict 2-box representation of H_1 and any three intervals of \mathcal{I}_5 corresponding to some vertices of $N \cup R$ are strictly overlapping, by Lemma 4 every (facial) triangle of H_1 has an empty inner corner in $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5)$. By Corollary 7, this representation of H_1 can be extended to T_1 , so that each vertex u of T_1 , lying inside some facial triangle xyz of H , is mapped to a 3-box B_u such that all the points of B_u are at L_∞ -distance at most $\frac{1}{4}$ from the empty inner corner of xyz . This shows how to extend $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5)$ to T_1 (in the sense that the restriction of $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5)$ to $N \cup R \cup T_1$ is a representation of the subgraph of G induced by $N \cup R \cup T_1$).

Similarly, consider the 3-box representation $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$, restricted to H_2 (i.e. to the vertices of $C \cup N$). As before, we can prove using Lemma 4 that all facial triangles of H_2 have an empty inner corner in this representation, and it follows from Corollary 7 that this representation of H_2 can be extended to T_2 , so that each vertex u of T_2 , lying inside some facial triangle xyz of H , is mapped to a 3-box B_u such that all the points of B_u are at L_∞ -distance at most $\frac{1}{4}$ from the empty inner corner of xyz . Recall that by definition of T_2 , such a triangle xyz intersects C , and therefore xyz is also a facial triangle of H_2 . This shows how to extend $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$ to T_2 (in the sense that the restriction of $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$ to $C \cup N \cup T_2$ is a representation of the subgraph of G induced by $C \cup N \cup T_2$) and completes the description of our 5-box representation $\mathcal{R} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$ of G .

We now prove that \mathcal{R} is a representation of G , i.e. G is precisely the intersection of the interval graphs \mathcal{I}_i , for $1 \leq i \leq 5$. By the definition of \mathcal{R} , the restriction of \mathcal{R} to $N \cup R \cup T_1$ represents the subgraph of G induced by $N \cup R \cup T_1$ and the restriction of \mathcal{R} to $C \cup N \cup T_2$ represents the subgraph of G induced by $C \cup N \cup T_2$. So we only need to consider pairs of vertices $u \in C \cup T_2$ and $v \in R \cup T_1$ (by definition, two such vertices u, v are non-adjacent in G), and show that the 5-boxes of u and v in \mathcal{R} are disjoint. To see this, consider only \mathcal{I}_5 and observe that all vertices of T_2 are mapped to intervals that are at distance at most $\frac{1}{4}$ of an interval of C , so all the intervals of $C \cup T_2$ end before $2 + \frac{1}{4} + \frac{1}{4} = \frac{5}{2}$. Similarly, all the intervals of $R \cup T_1$ start after $3 - \frac{1}{4} = \frac{11}{4}$. It follows that the intervals of u and v are disjoint in \mathcal{I}_5 and consequently, the boxes of u and v are disjoint in \mathcal{R} , as desired. \square

It was proved in [3] that toroidal graphs have boxicity at most 7, while there are toroidal graphs of boxicity 4. The proof was mainly based on the the following result of Schrijver [12]:

Theorem 9. [12] *Every graph embedded in the torus with face-width k contains $\lfloor 3k/4 \rfloor$ vertex-disjoint (and homotopic) non-contractible cycles.*

Using some ideas of the proof of Theorem 8, we now improve the bound on the boxicity of toroidal graphs from 7 to 6. Theorem 8 implies that graphs embedded on the torus with edge-width at least 40 have boxicity at most 5. We also decrease this bound on the edge-width from 40 to 4.

Theorem 10. *If G is a graph embedded on the torus, then G has boxicity at most 6. If, moreover, G has edge-width at least 4, then G has boxicity at most 5.*

Proof. Let G be a graph embedded on the torus. As before, we can assume without loss of generality that G is a triangulation, and thus the face-width and the edge-width of G are equal. If G has edge-width at least 6, then by Theorem 9, G has 4 vertex-disjoint (and homotopic) non-contractible cycles. Let C be one of them. We can assume that C is chordless (see [3]). Note that because of the 4 vertex-disjoint cycles, the subgraph of G induced by C and its neighborhood can be embedded in a cylinder, and is therefore planar. The exact same proof as that of Theorem 8 (with $g = 1$) then shows that G has boxicity at most 5.

Assume now that G has edge-width at most 5 and at least 4, and let C be a shortest non-contractible cycle (in particular, C is chordless). Observe that $G - C$ is planar; in the remainder of the proof, we fix a planar embedding of $G - C$. Since G has no non-contractible triangles, all the triangles of G (and $G - C$) are contractible. Let H be the frame of (the fixed embedding of) $G - C$. Let R be the set of vertices of H , and let T be the set of vertices of G lying in a non-facial triangle of (the fixed planar embedding of) $G - C$. If one of the two faces of $G - C$ with vertex-set $L(C)$ or $R(C)$ is contained in the interior of some triangle uvw of $G - C$, then uvw is homotopic to C in G , and thus non-contractible (which contradicts the fact that G has edge-width at least 4). It follows that $L(C)$ and $R(C)$ are included in R and consequently, no vertex of T is adjacent to a vertex of C . Let $(\mathcal{I}_1, \mathcal{I}_2)$ be a strict 2-box representation of $H = G[R]$ as in the proof of Theorem 8 (since G has edge-width at least 4, a proof similar to that of Theorem 8 shows that such a representation exists).

We first extend this 2-dimensional representation to C by mapping all the vertices of C to a large 2-box containing all the 2-boxes of $(\mathcal{I}_1, \mathcal{I}_2)$. Recall that C induces a cycle of length 4 or 5. We now partition the vertices of C into three independent sets S_3, S_4, S_5 , each containing one or two vertices. For any vertex v of G , let N_v denote the neighborhood of v in $C \cup R$. For each $3 \leq i \leq 5$, we denote by \mathcal{I}_i the interval graph with vertex-set $C \cup R$ depicted in Figure 4 (left) if $S_i = \{x\}$ or Figure 4 (right) if $S_i = \{x, y\}$. Observe that in $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$, C induces a cycle, R induces a clique, and the adjacency between C and R is the same as in G .

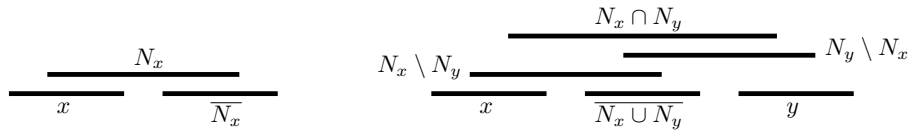


FIGURE 4. The description of \mathcal{I}_i with $S_i = \{x\}$ (left), and the description of \mathcal{I}_i with $S_i = \{x, y\}$ (right). We use the following notation to avoid overloading the figure: $\overline{N_x} = C \cup R \setminus (N_x \cup \{x\})$ (left) and $\overline{N_x \cup N_y} = C \cup R \setminus (N_x \cup N_y \cup \{x, y\})$ (right).

Since C contains at most 5 vertices, one of the sets S_i (say S_3) contains only one vertex, call it x . As before, we choose a small real $\epsilon_v > 0$ for each vertex v of H , so that all the chosen ϵ are distinct, and we change each interval $[s_v, t_v]$ of v in \mathcal{I}_3 to $[s_v + \epsilon_v, t_v + \epsilon_v]$. If each ϵ_v is small enough, this does not change the graph induced by \mathcal{I}_3 (and thus the graph induced by $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$). Let uvw be a triangle of H . Since uvw is disjoint from x (the unique

vertex of S_3), it follows from the definition of \mathcal{I}_3 (modified with the ϵ_v) that the intervals corresponding to u, v, w in \mathcal{I}_3 are strictly overlapping. By Lemma 4, every triangle of H has an empty inner corner in the strict 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$. By Corollary 7, we can extend the strict 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ of H to T (so that in \mathcal{I}_3 , all the newly added intervals are disjoint from the interval of x). In $(\mathcal{I}_4, \mathcal{I}_5)$, it remains to map all the vertices of T to a 2-box of $(\mathcal{I}_4, \mathcal{I}_5)$ intersecting all the 2-boxes except that of the vertices of C . This can be done for instance by mapping in \mathcal{I}_i each vertex of T to the interval labelled $\overline{N_x}$ in Figure 4 (left), or the interval labelled $\overline{N_x \cup N_y}$ in Figure 4 (right). This ensures that in $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$, the vertices of T are adjacent to all the other vertices of $G - C$, and non-adjacent to all the vertices of C , as desired. The representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$ of G then shows that G has boxicity at most 5.

Assume now that G has edge-width at most 3. Then a set S of at most 3 vertices can be removed from G so that $G - S$ is planar, and thus has boxicity at most 3. Take a 3-box representation of $G - C$, and extend it to C by mapping all the vertices of C to a large 3-box containing all the 3-boxes of the representation. Now add 3 intervals graphs, one for each element of S , defined as in Figure 4 (left), where we now define N_x , with $x \in S$, as the neighborhood of x in G , and $\overline{N_x}$ as $V(G) \setminus (N_x \cup \{x\})$. The obtained 6-box representation induces G , so G has boxicity at most 6. \square

4. LINEAR EXTENDABILITY

Theorem 8 easily implies that there exists a function f , such that for any $g \geq 0$ and any graph G embedded on \mathbb{S}_g , a set of at most $f(g)$ vertices can be removed from G so that the resulting graph has boxicity at most 5. However, the function f derived from Theorem 8 is exponential in g . In this section, we show how to make f linear in g . Note that the proof works for graphs of Euler genus g (while Theorem 8 is only concerned with graphs embeddable on \mathbb{S}_g). The previously best known result of this type was that $O(g)$ vertices can be removed in any graph of Euler genus g , so that the resulting graph has boxicity at most 42 [3].

We will use a technique of Kawarabayashi and Thomassen [7], who used it to prove several results of this type. Kawarabayashi and Thomassen [7, Theorem 1] proved that any graph G embedded on some surface of Euler genus g , with face-width more than $10t$ (for some constant t) has a partition of its vertex-set into three parts A, P, X , such that X has size at most $10tg$, P consists of the disjoint union of paths that are local geodesics (in the sense that each subpath with at most t vertices of a path of P is a shortest path in G and any two vertices at distance at least t in some path of P are at distance at least t in G) and are pairwise at distance at least t in G , and A induces a planar graph having a plane embedding H such that the only vertices of A having a neighbor in P lie on the outerface of H .

We will also use the following technical lemma.

Lemma 11. *Let G be a graph whose vertex-set is partitioned into two sets K and P , such that K induces a complete graph, P induces a path, and for every vertex u of K , the neighbors of u in P lie in a subpath of P of at most 3 vertices (equivalently, any two neighbors of u in P are at distance at most two in P). Then for any real number t , G has a 3-box representation*

$(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ such that all the intervals of \mathcal{I}_3 corresponding to some vertex of K end at t , while all the intervals of \mathcal{I}_3 corresponding to some vertex of P end strictly before t .

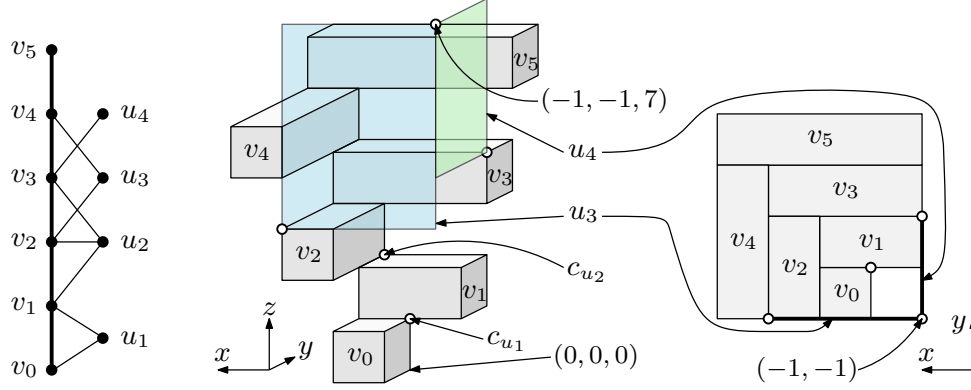


FIGURE 5. The point $(-1, -1, 7)$ and the bottom corners c_{u_i} of the vertices u_i are depicted with white dots. For the sake of readability, the 3-boxes of u_1 and u_2 are not displayed (only two of their corners are depicted).

Proof. We first construct a 3-box representation of G , and then show how to slightly modify it so that it satisfies the additional constraint on \mathcal{I}_3 .

Let $P = v_0, v_1, \dots, v_p$. For every $i \geq 0$, v_{2i} is mapped to the 3-box $[i, i+1] \times [-1, i] \times [2i, 2i+1]$ and v_{2i+1} is mapped to the 3-box $[-1, i+1] \times [i, i+1] \times [2i+1, 2i+2]$ (see Figure 5, where both a 3-dimensional view and a 2-dimensional view from above are depicted for the sake of clarity). Let u be a vertex of K . Then u is mapped to the 3-box with corners $(-1, -1, p+2)$ and c_u , where c_u is defined as follows. If u has no neighbor in P , then $c_u = (-1, -1, p+2)$. If u has a single neighbor v_j in P , then either $j = 2i$ and we define $c_u = (i, -1, 2i+1)$, or $j = 2i+1$ and we define $c_u = (-1, i, 2i+2)$ (see for example the 3-box of u_4 in Figure 5). If the neighbors of u are two consecutive vertices of P , say v_{2i} and v_{2i+1} (the case v_{2i+1}, v_{2i+2} can be handled analogously by switching the roles of the x - and y -axis), then we set $c_u = (i, i, 2i+1)$ (see for example the bottom corner c_{u_1} of u_1 in Figure 5). If the neighbors of u in P are v_{2i} and v_{2i+2} , then $c_u = (i, -1, 2i+1)$ (see for example the 3-box of u_3 in Figure 5). The case where the neighbors of u in P are v_{2i+1} and v_{2i+3} is handled analogously by switching the roles of the x - and y -axis. Finally, if the neighbors of u in P are $v_{2i}, v_{2i+1}, v_{2i+2}$, for some i , then we set $c_u = (i+1, i, 2i+1)$. Again, the case where the neighbors of u in P are $v_{2i+1}, v_{2i+2}, v_{2i+3}$ is handled analogously by switching the roles of the x - and y -axis (see for example the bottom corner c_{u_2} of u_2 in Figure 5).

All the 3-boxes of the vertices of K contain the point $(-1, -1, p+2)$, so K induces a complete graph in the representation defined above. Moreover, it readily follows from the definition of the 3-boxes of the vertices v_i and the corners c_u that for each vertex u of K , the neighbors of u in P are precisely the same in the graph G and in the 3-box representation defined above. Consequently, the constructed representation induces G , as desired.

Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ be the three interval graphs corresponding respectively to the x -, y -, and z -axis in the representation above. It follows from the construction of \mathcal{I}_3 that all the vertices

$v \in K$ are mapped in \mathcal{I}_3 to an interval of the form $[i_v, p + 2]$, while all the vertices of P are mapped in \mathcal{I}_3 to intervals ending strictly before $p + 2$. It is then easy to translate the whole representation along the z -axis so that it satisfies the additional property. \square

Finally, we will need the following direct consequence of Theorem 6.

Corollary 12. *Let G be a planar graph, let w be a fixed vertex of G , and let $t \in \mathbb{R}$. Then G has a strict 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ such that the interval I_w of w in \mathcal{I}_3 ends at t , while all the other intervals of \mathcal{I}_3 are contained in $[t, +\infty)$.*

Proof. We can assume without loss of generality that G is a triangulation (since it is an induced subgraph of some triangulation), and that w belongs to the outerface of G . We first map the three vertices of the outerface to 3-boxes as in Figure 2, and then apply Theorem 6 to extend this representation to a strict 3-box representation of G , such that the boxes of all the internal vertices are inside the inner corner C . Note that some hyperplane separates the box B_w of w from the boxes of all the other vertices of G , and the representation can therefore be translated in \mathbb{R}^3 in order to satisfy the desired property. \square

We are now able to prove the main result of this section.

Theorem 13. *Let G be a graph of Euler genus $g > 0$. Then G contains a set X of at most $60g - 30$ vertices such that $G - X$ has boxicity at most 5.*

Proof. We prove the theorem by induction on $g > 0$. If G has face-width at most 30, then G contains a set X of at most 30 vertices such that $G - X$ has Euler genus at most $g - 1$, or $G - X$ is the disjoint union of two graphs of Euler genus $g_1 > 0$ and $g_2 > 0$ with $g_1 + g_2 = g$ (see Proposition 4.2.1 and Lemma 4.2.4 in [9]). In the first case, either $G - X$ is planar (in which case the result clearly holds, since $G - X$ has boxicity at most 3 and $60g - 30 \geq 30$), or by the induction, a set X' of at most $30 + 60(g - 1) - 30 \leq 60g - 30$ can be removed from G in order to obtain a graph with boxicity at most 5. In the second case, by the induction, a set X' of at most $30 + (60g_1 - 30) + (60g_2 - 30) \leq 60g - 30$ can be removed from G in order to obtain a graph with boxicity at most 5. As a consequence, we can assume that G has face-width at least 30, and apply the result of Kawarabayashi and Thomassen mentioned above, with $t = 3$.

Let A, P, X be the corresponding partition of the vertex-set of G (and let H be the planarly embedded subgraph of G induced by A , such that only the outerface O of H has neighbors in P). Note that X contains at most $30g \leq 60g - 30$ vertices, and we will prove that $G - X = G[P \cup A]$, the subgraph of G induced by A and P , has boxicity at most 5.

Let H^+ be the planar graph obtained from H by adding a new vertex v^+ adjacent to all the vertices of O . By Corollary 12 (with $w = v^+$), H^+ has a strict 3-box representation $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3^+)$ such that for some real number p^+ , all the intervals of \mathcal{I}_3^+ corresponding to some vertex of O start at p^+ , while all the intervals of \mathcal{I}_3^+ corresponding to some vertex of $A \setminus O$ start (strictly) after p^+ .

Let v be a vertex of H . Since the paths of P are local geodesics, and any two paths are at distance at least 3 apart, v has at most 3 neighbors in P and these neighbors lie on a subpath of at most 3 vertices of a path of P (i.e. they are either consecutive or at distance two on some path of P). Let P_1, P_2, \dots, P_k be the paths of P , and for each $1 \leq i \leq k$,

consider the two endpoints of P_i and decide arbitrarily which one is the left endpoint and which one is the right endpoint. Let \tilde{H} be the graph obtained from $G[P \cup O]$ by adding, for each $1 \leq i < k$, a vertex v_i adjacent (only) to the right endpoint of P_i and the left endpoint of P_{i+1} , and by adding an edge between any two (non-adjacent) vertices of O . By Lemma 11, \tilde{H} has a 3-box representation $(\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_3^-)$ such that the intervals of \mathcal{I}_3^- either end at p^+ (if they correspond to a vertex of O), or end strictly before p^+ (if they correspond to a vertex of P). The restriction of $(\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_3^-)$ to $P \cup O$ induces a 3-box representation of the graph obtained from $G[P \cup O]$ by adding an edge between any two (non-adjacent) vertices of O , with \mathcal{I}_3^- satisfying the same additional property as above.

Let \mathcal{I}_3 be the interval representation obtained from \mathcal{I}_3^+ and \mathcal{I}_3^- as follows. Every vertex of $A \setminus O$ is mapped to its image in \mathcal{I}_3^+ , every vertex of P is mapped to its image in \mathcal{I}_3^- , and every vertex of O is mapped to the concatenation of its images in \mathcal{I}_3^- and \mathcal{I}_3^+ (note that the former ends at p^+ and the latter starts at p^+). Note that the adjacency between O and P is the same in \mathcal{I}_3 and \mathcal{I}_3^- , and the adjacency between O and $A \setminus O$ is the same in \mathcal{I}_3 and \mathcal{I}_3^+ . Moreover, the intervals of $A \setminus O$ are disjoint from the intervals of P in \mathcal{I}_3 .

It remains to map every vertex of P in $(\mathcal{I}_1, \mathcal{I}_2)$ to a large 2-box containing all the other 2-boxes of $(\mathcal{I}_1, \mathcal{I}_2)$, and to map every vertex of $A \setminus O$ in $(\mathcal{I}_4, \mathcal{I}_5)$ to a large 2-box containing all the other 2-boxes of $(\mathcal{I}_4, \mathcal{I}_5)$. Let $\mathcal{R} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$. Note that the restrictions to A of \mathcal{R} , $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$, and $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3^+)$, all induce the same graph (namely, $G[A]$). Similarly, the restrictions to $O \cup P$ of \mathcal{R} , $(\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5)$, and $(\mathcal{I}_3^-, \mathcal{I}_4, \mathcal{I}_5)$, all induce the same graph (namely, $G[O \cup P]$). Since the intervals of $A \setminus O$ are disjoint from the intervals of P in \mathcal{I}_3 , there are no edges between P and $A \setminus O$ in \mathcal{R} . It follows that \mathcal{R} represents $G[A \cup P] = G - X$, as desired. \square

5. LARGE GIRTH GRAPHS

The previous sections were devoted to graphs embedded in fixed surfaces, without short non-contractible cycles. Here we consider graphs without short cycles at all. Using the results of [1] relating the boxicity of a graph and its second largest eigenvalue (in absolute value), together with the existence of Ramanujan graphs of arbitrarily large degree and girth [8], it directly follows that there is a constant $c > 0$ such that for any integers d and g , there is a k -regular graph ($k \geq d$) of girth at least g and boxicity at least $ck/\log k$. As a consequence, there are (regular) graphs with arbitrarily large girth and boxicity. Therefore, in order to bound the boxicity of graphs without short cycles, it is necessary to restrict ourselves to specific classes of graphs.

Let $p \geq 1$ be an integer. A graph G is said to be *p-path-degenerate* (see [10]) if any subgraph H of G contains a vertex of degree at most 1, or a path with p internal vertices, each having degree two in H .

A 3-box representation of a graph G is called a *3-segment representation* if (1) each vertex is mapped to a *segment*, (the cartesian product of two points and an interval of positive length), (2) the interiors of any two segments are disjoint (in other words, the interior of a segment is only intersected by endpoints of other segments), and (3) no two segments lie on the same line. We will prove the following result:

Theorem 14. *Any 5-path-degenerate graph G has a 3-segment representation.*

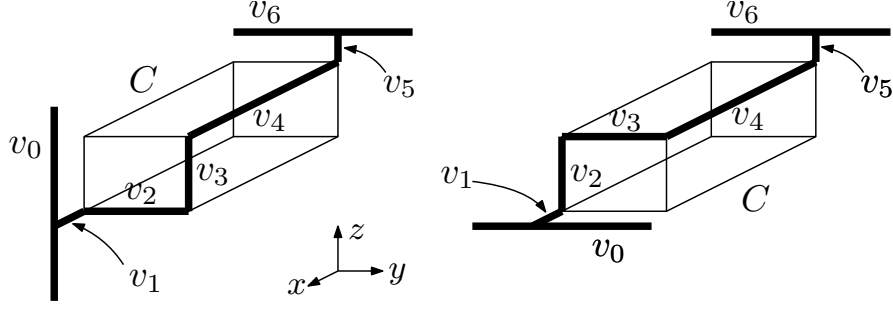


FIGURE 6. The representation of a path with 5 internal vertices of degree 2 between v_0 and v_6 when S_{v_0} and S_{v_6} are parallel to different axes (left) and when S_{v_0} and S_{v_6} are parallel to the same axis (right).

Proof. We will prove the result by induction on the number of vertices of G . Assume first that G contains a vertex v of degree at most 1, and let $H = G - v$. Note that H is 5-path-degenerate, so by the induction it has a 3-segment representation $\mathcal{S} = (S_v)_{v \in H}$. If v has degree 0, then G is the disjoint union of H and $\{v\}$, and clearly has a 3-segment representation. Thus, we can assume that v has a unique neighbor u in G . Since \mathcal{S} contains a finite number of segments, S_u contains a point p such that some small ball B centered in p only intersects S_u . We then represent S_v as a segment orthogonal to S_u (there are two possible choices of dimension), with p as one endpoint, and such that S_v lies inside B .

In the remainder of the proof, we assume that G contains a path $P = v_0v_1 \dots v_6$, such that for any $1 \leq i \leq 5$, the only neighbors of v_i in G are v_{i-1} and v_{i+1} . Let H be the graph obtained from G by removing all the vertices v_i with $1 \leq i \leq 5$, and let $\mathcal{S} = (S_v)_{v \in H}$ be a 3-segment representation of H . We now extend \mathcal{S} to the vertices v_1, v_2, \dots, v_5 . For $i = 0, 6$, fix a point $p_i \in S_{v_i}$ such that some small ball B_i centered in p_i only intersects S_{v_i} . Assume without loss of generality that either S_{v_0} is parallel to the z -axis and S_{v_6} is parallel to the y -axis (this includes the case where v_0 and v_6 are adjacent), or both S_{v_0} and S_{v_6} are parallel to the y -axis (this includes the case where v_0 and v_6 are the same vertex). Assume that B_0 and B_6 have radius at least ϵ , for some $\epsilon > 0$. We map v_1 to the segment S_{v_1} that has p_0 as an endpoint, is parallel to the x -axis, has length ϵ , and goes in the direction of p_6 (parallel to the x -axis). Similarly, we map v_5 to the segment S_{v_5} that has p_6 as an endpoint, is parallel to the z -axis, has length ϵ , and goes in the direction of p_0 (parallel to the z -axis). Let p_1 be the endpoint of S_{v_1} distinct from p_0 , and let p_5 be the endpoint of S_{v_5} distinct from p_6 . Let C be the 3-box with corners p_1 and p_5 . We map v_2 to the edge S_{v_2} of C which contains p_1 and is orthogonal to S_{v_0} and S_{v_1} , and similarly we map v_4 to the edge S_{v_4} of C which contains p_5 and is orthogonal to S_{v_5} and S_{v_6} . Finally, we map v_3 to the edge S_{v_3} of C connecting the endpoint of S_{v_2} distinct from p_1 and the endpoint of S_{v_4} distinct from p_5 (see Figure 6). Note that the description of S_{v_i} , $1 \leq i \leq 5$, above only depends of the choice of p_0 , p_1 , and ϵ . We now move each of p_0 , p_1 and the value of ϵ along a tiny interval. Then the locus described by each S_{v_i} , $1 \leq i \leq 5$, is a non-degenerate 3-box. Since \mathcal{S} is the union of a finite number of segments, it follows that we can choose p_0 , p_1 , and $\epsilon > 0$ so that the S_{v_i} ($1 \leq i \leq 5$) are disjoint from \mathcal{S} . We can moreover choose p_0 , p_1 , and $\epsilon > 0$ so that C

is a non-degenerate 3-box (and so the S_{v_i} are (non-degenerate) segments) and no segment S_{v_i} lie on the same line as a segment of \mathcal{S} . Consequently, the obtained representation is a 3-segment representation and each vertex v_i with $1 \leq i \leq 5$ is mapped to a segment that only intersects the segments of v_{i-1} and v_{i+1} , as desired. \square

It was proved by Galluccio, Goddyn and Hell [5] that for any proper minor-closed class \mathcal{F} and for any k , there is an integer $g = g(k)$ such that any graph of \mathcal{F} with girth at least g is k -path-degenerate. Since 3-segment representations are 3-box representations, we have the following immediate consequence.

Corollary 15. *For any proper minor-closed class \mathcal{F} there is an integer $g = g(\mathcal{F})$ such that any graph of \mathcal{F} of girth at least g has boxicity at most 3.*

Note that the result of Galluccio, Goddyn and Hell was recently extended by Nešetřil and Ossona de Mendez [10] to classes of subexponential expansion, i.e. expansion bounded by $d \mapsto \exp(d^{1-\epsilon})$, for some $\epsilon > 0$ (see [10] for definitions and further details). This shows that Corollary 15 can be extended to this fairly broad setting as well.

Interestingly, the bound on the boxicity in Corollary 15 is best possible already for the class of K_6 -minor free graphs. The following example was given by Stéphan Thomassé. Take a copy of K_5 , the complete graph on 5 vertices, and replace each edge by an arbitrarily large path. The resulting graph has arbitrarily large girth, no K_6 -minor, and any 2-box representation of it would give a planar embedding (without crossings) of K_5 , a contradiction.

For graphs of Euler genus g , Theorem 3.2 in [5] (combined with Theorem 14) implies the following interesting counterpart of Theorem 8 (see the difference between the exponential bound there and the logarithmic bound here).

Corollary 16. *There is a constant c such that any graph of Euler genus g and girth at least $c \log g$ has boxicity at most 3.*

Observe that this is best possible up to the choice of the constant c : for any integer k , there is a constant $c' = c'(k)$ and an infinite family of graphs of (increasing) Euler genus g , girth at least $c' \log g$, and boxicity at least k . This follows from the results of [1] mentioned in the introduction of this section, and the fact that the Ramanujan graphs described in [8] have girth logarithmic in their number of vertices (and Euler genus linear in their number of vertices, at least for d -regular graphs with $d \geq 7$).

It is worth noting that Theorem 14 can also be applied to classes that do not fit well in the framework of Nešetřil and Ossona de Mendez [10] (because their density is too high). Examples of such classes include *segment* or *strings* graphs (intersection graphs of segments, or strings in the plane), or *circle* graphs (intersection graphs of chords of a circle). For example, it can be proved using Theorem 14 and the results of [4] that every circle graph of girth at least 9 has boxicity at most 3. This is in contrast with the existence of a circle graph (indeed, a permutation graph) on $2n$ vertices with boxicity n , for every $n \geq 1$ (see [11]).

6. CONCLUSION

A natural problem is to find a counterpart of Theorem 8 for non-orientable surfaces. Non-orientable versions of Theorem 1 exist [15], so the only problem when applying the same

arguments as in the proof of Theorem 8 to a graph embedded in a non-orientable surface is that some of the cycles in the planarizing collection might be one-sided, in which case the cycle C together with its neighborhood N does not necessarily embed in a cylinder, but instead on a Möbius strip. As a consequence, these graphs are not necessarily planar. However, using Lemma 11, we can prove that the graph obtained from the subgraph induced by $C \cup N$ by adding an edge between any two vertices of N , has boxicity at most 4 (just remove a vertex of C and apply Lemma 11). Consequently, a proof along the lines of the proof of Theorem 8 easily shows that locally planar graphs embedded on non-orientable surfaces have boxicity at most $4 + 2 = 6$.

It would be interesting to improve this bound, as well as that of Theorem 8. It was conjectured in [3] that locally planar graphs have boxicity at most 3 (which would be best possible since there are planar graphs of boxicity 3). We also conjecture the following variant:

Conjecture 17. *There is a constant $c > 0$ such that in every graph embedded on a surface of Euler genus g , at most cg vertices can be removed so that the resulting graph has boxicity at most 3.*

Note that the linear bound (in g) would be best possible, since there are toroidal graphs with boxicity 4 (for example K_8 minus a perfect matching, see [3]), and the disjoint union of $\Omega(g)$ such graphs can be embedded in a surface of Euler genus g .

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